$$D = \left(k^\circ - \frac{d^2}{5k^\circ}\right)\sqrt{\varepsilon_{ij}{}^\circ\varepsilon_{ij}{}^\circ} \tag{3.4}$$

The dissipative function (3.3) corresponds to the Mises plasticity condition. The plasticity limit can be expressed by means of given concentrations and plasticity limits of each component. In the case of a two-component medium the plasticity limit is calculated by means of the formula

$$k = c_1k_1 + c_2k_2 - \frac{c_1c_2(k_1 - k_2)^2}{5(c_1k_1 + c_2k_2)} \tag{3.5}$$

where c_1, c_2, k_1, k_2 are the concentrations and plasticity limits of the corresponding components.

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EXTENDED ORTHOGONALITY RELATIONSHIPS FOR SOME PROBLEMS OF ELASTICITY THEORY

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Orthogonality relationships are derived for the extended eigenvectors of problems on the deformation of a strip, a circular rectangle and the axisymmetric deformation of a cylinder under homogeneous boundary conditions in the displacements.

The problem of the simultaneous decomposition of the boundary conditions given on parts of the surface of an elastic body into a series of nonorthogonal homogeneous solutions is solved only for certain classical problems for definite combinations of the boundary conditions. In the case of the plane problem of the theory of elasticity for a strip, such decompositions are realizable because of the generalized orthogonality relationship of Papkovich [1–4]. A similar relationship for the axisymmetric problem of a cylinder is obtained in [5] and generalized in [6]. However, the mentioned orthogonality relationships do not allow satisfaction of arbitrary boundary conditions exactly on all surfaces of an elastic body of finite size.

Of interest in this respect are the orthogonality relationships of extended eigenvectors of boundary value problems. The elasticity theory equations admit the non-unique construction of such vectors. Thus, Little and Childs [7, 8] construct a system of extended eigenvectors which are orthogonal to the vectors of the conjugate problems. The authors called such orthogonality relationships biorthogonality.

The method developped in [9], which permits construction of a system of extended eigenvectors satisfying the self-adjoint differential equations and hence possessing the orthogonality property, is more natural and general. This method also yields a non-unique solution (the authors of the method did not use the best), however, only systems of such vectors whose projections correspond to combinations of quantities prescribed on the boundary (displacements, normal and shear stresses) are of practical interest.

Precisely such systems of extended vectors are constructed below by the Flügge-Kelkar method for three two-dimensional problems of elasticity theory. Since the method mentioned proceeds from the displacements equations, the case of homogeneous boundary conditions in displacements is simplest, and is indeed examined. However, the method can be extended also to the case of homogeneous force conditions.

1. Strip. Let us examine a strip whose longitudinal edges are fixed

$$u = v = 0 \qquad \text{for} \qquad y = \pm 1 \tag{1.1}$$

The initial equations in displacements for the plane state of stress are written as

$$2m \frac{\partial^2 u}{\partial x^2} + (m-1) \frac{\partial^2 u}{\partial y^2} + (m+1) \frac{\partial^2 v}{\partial x \, \partial y} = 0, \quad 2m \frac{\partial^2 v}{\partial y^2} + (m-1) \frac{\partial^2 v}{\partial x^2} + $$

$$+ (m+1) \frac{\partial^2 u}{\partial x \, \partial y} = 0 \tag{1.2}$$

Here u, v are the projections of the displacements on the x, y axes, respectively, m is the Poisson's ratio. Let us seek the solution of (1.2) in the form

$$\mathbf{u} = \left\| \begin{matrix} u \\ v \end{matrix} \right\| = \xi(y) e^{-\lambda x}, \qquad \xi(y) = \left\| \begin{matrix} f(y) \\ h(y) \end{matrix} \right\| \tag{1.3}$$

Substituting (1.3) into (1.2) yields a relationship to find the vectors $\xi(y)$

$$\xi'' = \lambda \mathbf{L}_1 \xi' + \lambda^2 \mathbf{L}_2 \xi \tag{1.4}$$

Here $\mathbf{L}_1$ and $\mathbf{L}_2$ are the matrices

$$\mathbf{L}_1 = \left\| \begin{matrix} 0 & l_{12}^{(1)} \\ l_{21}^{(1)} & 0 \end{matrix} \right\|, \qquad \mathbf{L}_2 = \left\| \begin{matrix} l_{11}^{(2)} & 0 \\ 0 & l_{22}^{(2)} \end{matrix} \right\|$$

$$l_{12}^{(1)} = \frac{m+1}{m-1}, \quad l_{21}^{(1)} = \frac{m+1}{2m}, \quad l_{11}^{(2)} = \frac{-2m}{m-1}, \quad l_{22}^{(2)} = \frac{1-m}{2m}$$

and the prime denotes differentiation with respect to y. The vector $\xi(y)$ should satisfy the homogeneous boundary conditions (1.1). The boundary value problem (1.4), (1.1) generates an infinite system of eigenvectors (*)

$$\xi_k(y, \lambda_k) = \left\| \begin{matrix} f_k(y, \lambda_k) \\ h_k(y, \lambda_k) \end{matrix} \right\| \qquad (k = 0, 1, 2, \ldots)$$

corresponding to the eigenvalues λ_k of the parameter λ, the roots of the equation

$$\frac{3m-1}{m+1} \sin 2\lambda \pm 2\lambda = 0$$

The system of eigenvectors $\xi_k(y, \lambda_k)$ does not possess the orthogonality property in the interval $(-1, 1)$.

Corresponding to the displacement vector (1.3), which now becomes

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*) The kind of eigenvectors of the problems considered here can be found in [7, 9, 10].

$$\mathbf{u} = \sum_{k=0}^{\infty} C_k \xi_k (y)\, e^{-\lambda_k x} \tag{1.5}$$

are the stresses

$$\frac{1}{\mu} \sigma_x = \sum_{k=0}^{\infty} C_k \sigma_k (y)\, e^{-\lambda_k x}, \quad \frac{1}{\mu} \tau_{xy} = \sum_{k=0}^{\infty} C_k \tau_k (y)\, e^{-\lambda_k x}$$

Here

$$\sigma_k = \frac{2}{m-1} (h_k' - m\lambda_k f_k), \qquad \tau_k = f_k' - \lambda_k h_k \tag{1.6}$$

Let us construct the system of extended vectors

$$\mathbf{z} = \begin{Vmatrix} \xi (y) \\ \eta (y) \end{Vmatrix}, \qquad \eta (y) = \begin{Vmatrix} p (y) \\ q (y) \end{Vmatrix} \tag{1.7}$$

(the subscript k has been omitted). Let us find the additional vector $\eta (y)$ from the conditions

$$\xi' = \mathbf{P}\xi + \mathbf{Q}\eta, \qquad \eta' = \mathbf{S}\eta \tag{1.8}$$

Here $\mathbf{P}, \mathbf{Q}, \mathbf{S}$ are 2×2 matrices linearly dependent on λ.

Let us note that according to conditions (1.8) we can write

$$\eta = \mathbf{Q}^{-1}\xi' - \mathbf{Q}^{-1}\mathbf{P}\xi \tag{1.9}$$

The superscript -1 denotes the inverse matrix.

Let us require that the elements of the vector $\eta (y)$, the functions $p (y)$, $q (y)$, correspond to combinations of conditions assigned on the strip boundaries $x = \text{const}$ (displacements, their derivatives with respect to y, normal and shear stresses). Then there necessarily results from (1.9)

$$\mathbf{P} = \lambda \mathbf{P}_1, \quad \mathbf{Q} = \mathbf{Q}_0, \quad \mathbf{S} = \lambda \mathbf{S}_1$$

The matrices $\mathbf{P}_1$, $\mathbf{Q}_0$, $\mathbf{S}_1$ are independent of λ. Substituting (1.8), (1.9) into (1.4) yields a system of equations to find the matrices $\mathbf{P}_1$, $\mathbf{Q}_0$, $\mathbf{S}_1$. We have

$$\mathbf{P}_1 + \mathbf{Q}_0\mathbf{S}_1\mathbf{Q}_0^{-1} - \mathbf{L}_1 = 0, \qquad \mathbf{Q}_0\mathbf{S}_1\mathbf{Q}_0^{-1}\mathbf{P}_1 + \mathbf{L}_2 = 0 \tag{1.10}$$

The matrices

$$\mathbf{P}_1 = \mathbf{S}_1 = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}, \qquad \mathbf{Q}_0 = \begin{Vmatrix} -1 & 0 \\ 0 & q_{22}^{(0)} \end{Vmatrix}, \qquad q_{22}^{(0)} = \frac{m-1}{2m}$$

are the solution of Eqs. (1.10).

We now have according to (1.6), (1.7), (1.9)

$$\begin{aligned} p_k (y) &= -f_k' - \lambda_k h_k = \tau_k - 2f_k' \\ q_k (y) &= \frac{2m}{m-1} (h_k' - \lambda_k f_k) = \sigma_k + 2h_k' \end{aligned} \tag{1.11}$$

As follows from (1.8), the extended vectors (1.7) satisfy the equations of the following boundary value problem:

$$\mathbf{z}' = \mathbf{A}\mathbf{z} + \lambda \mathbf{B}\mathbf{z}, \qquad \mathbf{M}\mathbf{z} (\pm 1) = 0 \tag{1.12}$$

Here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$ are the matrices

$$\mathbf{A} = \begin{Vmatrix} 0 & \mathbf{Q}_0 \\ 0 & 0 \end{Vmatrix}, \qquad \mathbf{B} = \begin{Vmatrix} \mathbf{P}_1 & 0 \\ 0 & \mathbf{S}_1 \end{Vmatrix}, \qquad \mathbf{M} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix} \tag{1.13}$$

Let us show that the problem (1.12) is self-adjoint. To do this it is necessary [11] that there exist a nondegenerate transformation

$$\mathbf{w} = \mathbf{T}\mathbf{z} \tag{1.14}$$

which will transform the conjugate boundary value problem

$$\mathbf{w}' = -\mathbf{A}^*\mathbf{w} - \lambda\mathbf{B}^*\mathbf{w}, \qquad \mathbf{N}\mathbf{w}(\pm 1) = 0 \tag{1.15}$$

into that under investigation (1.12). The superscript * denotes the transpose of the matrix.

This requirement yields three equations

$$\mathbf{TA} + \mathbf{A}^*\mathbf{T} = 0, \quad \mathbf{TB} + \mathbf{B}^*\mathbf{T} = 0, \quad \mathbf{MT}^{-1}\mathbf{M}^* = 0 \tag{1.16}$$

The matrix

$$\mathbf{T} = \begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}$$

is the simplest solution of the system (1.16).

Let us find the orthogonality relationship of the vectors $\mathbf{z}_k(y)$. By multiplying (1.12) for the vector $\mathbf{z}_n$ by the vector $\mathbf{w}_m^*$ on the left, and the transposed equation (1.15) for $\mathbf{w}_m$ by $\mathbf{z}_n$ on the right, adding, and integrating the result between -1 and 1, we substitute the transformation (1.14). We obtain

$$(\lambda_n - \lambda_m)\int_{-1}^{1} \mathbf{z}_m^*\mathbf{TB}\mathbf{z}_n\,dy = [\mathbf{z}_m^*\mathbf{T}\mathbf{z}_n]_{-1}^{1}$$

Taking account of (1.7), (1.11) and the boundary conditions (1.1), we have the orthogonality relationship for the vectors $\mathbf{z}_k(y)$

$$\int_{-1}^{1} \mathbf{z}_m^*\mathbf{R}\mathbf{z}_n\,dy = 0, \qquad n \neq m \tag{1.17}$$

Here $\mathbf{R}$ is the weight matrix

$$\mathbf{R} = \mathbf{TB} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix}$$

Expanding the relationship (1.17) by using (1.3), (1.11), we write it in the different form

$$\int_{-1}^{1} (f_m\sigma_n - \tau_m h_n + f_n\sigma_m - \tau_n h_m)\,dy = 0, \quad n \neq m \tag{1.18}$$

The orthogonality relationships (1.17), (1.18) permits finding the coefficients of the decomposition of the arbitrary vector $\bar{\mathbf{z}}_0$

$$\bar{\mathbf{z}}_0^* = \|\bar{f}_0(y), \bar{h}_0(y), \bar{p}_0(y), \bar{q}_0(y)\|$$

in a series in the extended vectors

$$\bar{\mathbf{z}}_0 = \sum_{k=0}^{\infty} C_k\mathbf{z}_k(y)$$

$$C_k = \frac{1}{G_k}\int_{-1}^{1}(\bar{f}_0\sigma_k - \bar{\tau}_0 h_k + f_k\bar{\sigma}_0 - \tau_k\bar{h}_0)\,dy, \quad G_k = 2\int_{-1}^{1}(f_k\sigma_k - \tau_k h_k)\,dy$$

Here G_k are normalizing factors.

2. Circular rectangle. Let us consider the deformation of a circular rectangle whose arc edges are fixed

$$u = v = 0 \quad \text{for } \rho = a, \quad \rho = b \tag{2.1}$$

Here u, v are the projections of the displacement vector on the axes of the polar coordinates ρ, φ, respectively.

Let us transform to the new coordinate

$$t = \ln \frac{\rho}{\sqrt{ab}} \qquad (-h \leqslant t \leqslant h) \qquad \left(h = \frac{1}{2} \ln \frac{a}{b}\right)$$

The equations in displacements for the plane state of stress become

$$\begin{gathered} 2m \frac{\partial^2 u}{\partial t^2} + (m-1) \frac{\partial^2 u}{\partial \varphi^2} - 2mu + (m+1) \frac{\partial^2 v}{\partial t\, \partial \varphi} - (3m-1) \frac{\partial v}{\partial \varphi} = 0 \\ 2m \frac{\partial^2 v}{\partial \varphi^2} + (m-1) \frac{\partial^2 v}{\partial t^2} - (m-1)\, v + (m+1) \frac{\partial^2 u}{\partial t\, \partial \varphi} + (3m-1) \frac{\partial u}{\partial \varphi} = 0 \end{gathered} \tag{2.2}$$

Let us seek the solution of (2.2) in the form

$$\mathbf{u} = \left\| \begin{matrix} u \\ v \end{matrix} \right\| = \boldsymbol{\xi}(t)\, e^{-\lambda\varphi}, \qquad \boldsymbol{\xi}(t) = \left\| \begin{matrix} f(t) \\ h(t) \end{matrix} \right\| \tag{2.3}$$

Substituting (2.3) into (2.2) yields a relationship to find the vector $\boldsymbol{\xi}(t)$

$$\boldsymbol{\xi}'' = \lambda \mathbf{L}_1 \boldsymbol{\xi}' + \lambda^2 \mathbf{L}_2 \boldsymbol{\xi} + \lambda \mathbf{L}_3 \boldsymbol{\xi} + \mathbf{L}_4 \boldsymbol{\xi} \tag{2.4}$$

$$\mathbf{L}_1 = \left\| \begin{matrix} 0 & l_{12}^{(1)} \\ l_{21}^{(1)} & 0 \end{matrix} \right\|, \qquad \mathbf{L}_2 = \left\| \begin{matrix} l_{11}^{(2)} & 0 \\ 0 & l_{22}^{(2)} \end{matrix} \right\|, \qquad \mathbf{L}_3 = \left\| \begin{matrix} 0 & l_{12}^{(3)} \\ l_{21}^{(3)} & 0 \end{matrix} \right\|$$

$$\mathbf{L}_4 = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|, \quad \begin{matrix} l_{12}^{(1)} = \dfrac{m+1}{2m}, & l_{21}^{(1)} = \dfrac{m+1}{m-1}, & l_{11}^{(2)} = \dfrac{1-m}{2m} \\ l_{22}^{(2)} = \dfrac{-2m}{m-1}, & l_{12}^{(3)} = \dfrac{1-3m}{2m}, & l_{21}^{(3)} = \dfrac{3m-1}{m-1} \end{matrix}$$

The primes denote differentiation with respect to t.

The vector $\boldsymbol{\xi}(t)$ should satisfy the boundary conditions (2.1) besides the Eq. (2.4). The boundary value problem (2.1), (2.14) has an infinite system of nonorthogonal eigenvectors

$$\boldsymbol{\xi}_k(t, \lambda_k) = \left\| \begin{matrix} f_k(t, \lambda_k) \\ h_k(t, \lambda_k) \end{matrix} \right\| \qquad (k = 0, 1, 2 \ldots)$$

corresponding to the eigenvalues λ_k of the parameter λ, the roots of the equation

$$(m+1)^2 (\lambda^2 + 1) \operatorname{sh}^2 2h + (3m-1)^2 \operatorname{sh} 2(\lambda i + 1) h \operatorname{sh} 2(\lambda i - 1) h = 0 \tag{2.5}$$

Corresponding to the displacement vector (2.3) which now becomes

$$\mathbf{u} = \sum_{k=0}^{\infty} C_k \boldsymbol{\xi}_k(t)\, e^{\lambda_k \varphi}$$

are the stresses

$$\frac{1}{\mu} \rho \sigma_\varphi = \sum_{k=0}^{\infty} C_k \sigma_k(t)\, e^{-\lambda_k \varphi}, \qquad \frac{1}{\mu} \rho \tau_{\rho\varphi} = \sum_{k=0}^{\infty} C_k \tau_k(t)\, e^{-\lambda_k \varphi} \tag{2.6}$$

Let us construct the system of extended vectors of the problem. Omitting calculations analogous to those presented in the problem for the strip, let us write the result

$$\mathbf{z}_k^* = \| f_k,\ h_k,\ p_k,\ q_k \| \quad (k = 0, 1, 2 \ldots) \tag{2.7}$$

where in the notation (2.6)

$$p_k = \frac{2m}{m-1}(f_k' + f_k - \lambda_k h_k) = \sigma_k + 2f_k'$$
$$q_k = -h_k' - h_k - \lambda_k f_k = \tau_k - 2h_k' \tag{2.8}$$

It is easy to verify that the vectors $\mathbf{z}_k(t)$ satisfy the self-adjoint differential equation and boundary conditions

$$\mathbf{z}' = \mathbf{A}\mathbf{z} + \lambda\mathbf{B}\mathbf{z}, \qquad \mathbf{M}\mathbf{z}(\pm h) = 0 \tag{2.9}$$

Here $\mathbf{A}$, $\mathbf{B}$, $\mathbf{M}$ are the matrices $(a_{13} = {}^1/_2\,(m-1)/m)$

$$\mathbf{A} = \begin{Vmatrix} -1 & 0 & a_{13} & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}, \quad \mathbf{B} = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}, \quad \mathbf{M} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{Vmatrix}$$

The orthogonality relationship of the vector $\mathbf{z}_k(t)$ is

$$\int_{-h}^{h} \mathbf{z}_m^* \mathbf{R} \mathbf{z}_n dt = 0 \quad (n \neq m), \qquad \mathbf{R} = \begin{Vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{Vmatrix} \tag{2.10}$$

Decomposing the relationship (2.10) by using (2.7), (2.8), we obtain

$$\int_{-h}^{h} (f_m \tau_n - \sigma_m h_n + f_n \tau_m - \sigma_n h_m)\, dt = 0 \quad (n \neq m) \tag{2.11}$$

Utilizing the orthogonality relationship (2.11), we find the coefficients of the expansion of the arbitraty vector $\bar{\mathbf{z}}^*_0 = \| \bar{f}_0(t), \bar{h}_0(t), \bar{p}_0(t), \bar{q}_0(t) \|$ in a series of eigenvectors of the problem (2.9)

$$\bar{\mathbf{z}}_0 = \sum_{k=0}^{\infty} C_k \mathbf{z}_k(t), \qquad C_k = \frac{1}{G_k} \int_{-h}^{h} (\bar{f}_0 \tau_k - \bar{\sigma}_0 h_k + f_k \bar{\tau}_0 - \sigma_k \bar{h}_0)\, dt$$

Here G_k are the normalizing factors

$$G_k = 2 \int_{-h}^{h} (f_k \tau_k - \sigma_k h_k)\, dt$$

3. Cylinder [9]. Let us examine the axisymmetric deformation of a hollow circular cylinder with axis x whose side surfaces $r = b$, $r = a$ $(a > b)$ are fixed

$$u = v = 0 \quad \text{for} \quad r = a, \ r = b \tag{3.1}$$

Here u, v are projections of the displacement vector $\mathbf{u}$ on the axis of the cylindrical x, r coordinates, respectively.

In this case the equations in displacements are

$$\frac{m-2}{2m}\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + \frac{m-1}{m}\frac{\partial^2 u}{\partial x^2} + \frac{1}{2}\left(\frac{\partial^2 v}{\partial x \partial r} + \frac{1}{r}\frac{\partial v}{\partial x}\right) = 0$$
$$\frac{1}{2}\frac{\partial^2 u}{\partial x \partial r} + \frac{m-1}{m}\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v^2}{r^2}\right) + \frac{m-2}{2m}\frac{\partial^2 v}{\partial x^2} = 0 \tag{3.2}$$

Setting

$$\mathbf{u} = \xi(r)\, e^{-\lambda x}, \qquad \xi(r) = \left\| \begin{matrix} f(r) \\ h(r) \end{matrix} \right\|$$

we obtain the equation

$$(r\xi')' = \lambda \mathbf{L}_1 \xi' + \lambda^2 \mathbf{L}_2 \xi + \lambda \mathbf{L}_3 \xi + \mathbf{L}_4 \xi \tag{3.3}$$

$$\mathbf{L}_1 = r \left\| \begin{matrix} 0 & l_{12}^{(1)} \\ l_{21}^{(1)} & 0 \end{matrix} \right\|, \quad \mathbf{L}_2 = r \left\| \begin{matrix} l_{11}^{(2)} & 0 \\ 0 & l_{22}^{(2)} \end{matrix} \right\|, \quad \mathbf{L}_3 = \left\| \begin{matrix} 0 & l_{12}^{(3)} \\ 0 & 0 \end{matrix} \right\|, \quad \mathbf{L}_4 = \frac{1}{r} \left\| \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right\|$$

$$l_{12}^{(1)} = \frac{m}{m-2}, \quad l_{21}^{(1)} = \frac{m}{2(m-1)}, \quad l_{11}^{(2)} = -\frac{2(m-1)}{m-2}, \quad l_{22}^{(2)} = -\frac{m-2}{2(m-1)}$$

$$l_{12}^{(3)} = \frac{m}{m-2}$$

which, together with the boundary conditions (3.1), generates the system of eigenvectors $\xi_k(r)$, nonorthogonal in the interval (b, a). Now

$$\mathbf{u} = \sum_{k=0}^{\infty} C_k \xi_k(r)\, e^{-\lambda_k x} \tag{3.4}$$

The stresses

$$\frac{1}{\mu}\sigma_x = \sum_{k=0}^{\infty} C_k \sigma_k(r)\, e^{-\lambda_k x}, \qquad \frac{1}{\mu}\tau_{rx} = \sum_{k=0}^{\infty} C_k \tau_k(r)\, e^{-\lambda_k x}$$

$$\sigma_k = \frac{2}{m-2}\left[h_k' + \frac{1}{r}h_k - (m-1)\lambda_k f_k\right], \qquad \tau_k = f_k' - \lambda_k h_k$$

correspond to the displacements (3.4).

A system of extended vectors

$$\mathbf{z}_k^*(r) = \| f_k,\ h_k,\ r(\tau_k - 2f_k'),\ r\sigma_k + 2(rh_k)' \| \quad (k = 0, 1, 2, \ldots) \tag{3.5}$$

satisfying the self-adjoint differential equation

$$r\mathbf{z}' = \left\| \begin{matrix} 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & m_{24} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{matrix} \right\| \mathbf{z} + \lambda r \left\| \begin{matrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{matrix} \right\| \mathbf{z}, \quad m_{24} = \frac{m-2}{2(m-1)} \tag{3.6}$$

and the boundary condition

$$\left\| \begin{matrix} \mathbf{E} & 0 \\ 0 & 0 \end{matrix} \right\| \mathbf{z} \binom{a}{b} = 0, \qquad \mathbf{E} = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\|$$

can be constructed by the method elucidated in Sect. 1.

The orthogonality relationship

$$\int_b^a \mathbf{z}_n^* \mathbf{R} \mathbf{z}_m\, dr = 0 \qquad (n \neq m), \qquad \mathbf{R} = \left\| \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} \right\| \tag{3.7}$$

is conserved for the vector (3.5).

Expanding the relationship (3.7) by using (3.5), we obtain

$$\int_b^a (f_m \sigma_n - \tau_m h_n + f_n \sigma_m - \tau_n h_m)\, r\, dr = 0, \quad n \neq m \tag{3.8}$$

The coefficients of the expansion

$$\bar{z}_0 = \sum_{k=0}^{\infty} C_k z_k(r)$$

of the arbitrary vector

$$\bar{z}_0^* = \| \bar{f}_0(r),\ \bar{h}_0(r),\ \bar{p}_0(r),\ \bar{q}_0(r) \|$$

are found because of the relationship (3.8) from

$$C_k = \frac{1}{G_k} \int_b^a (\bar{f}_0 \sigma_k - \bar{\tau}_0 h_k + f_k \bar{\sigma}_0 - \tau_k \bar{h}_0)\, r\, dr$$

where G_k are normalizing factors

$$G_k = 2 \int_b^a (f_k \sigma_k - \tau_k h_k)\, r\, dr$$

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